

Dining Cryptographers are Practical

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Abstract. The dining cryptographers protocol provides information-theoretically secure sender and recipient untraceability. However, the protocol is impractical because a malicious participant who disrupts the communication is hard to detect. We propose a scheme with optimal collision resolution, in which computationally limited disruptors are easy to detect.

1 Introduction

The Dining Cryptographers protocol [4] is a special primitive for anonymous communication in which senders and recipients are unconditionally untraceable. Unlike relay-based techniques like mixing or onion routing, it requires no assumption on the network, no cryptographic assumptions and no third party. Because of these advantages it could be useful in many scenarios like electronic voting, low latency anonymous communication or multiparty computation.

During a typical round of the protocol, the participants P_1, \dots, P_n respectively broadcast the ciphertexts O_1, \dots, O_n . Each ciphertext O_i looks like a random value, but the sum of all the ciphertexts $C = \sum_{i=1}^n O_i$ reveals an anonymous message M (i.e., $C = M$). The sender remains unknown; that is, each participant could be the sender of the message. The protocol typically comprises two steps:

1. During the first step, each pair of participants P_i and P_j secretly agrees on a key K_{ij} . This can be represented by a key graph like the one shown in Figure 1.1(a). By definition $K_{ji} = -K_{ij}$ and $K_{ii} = 0$.
2. During the second step, each participant P_i computes a ciphertext O_i by computing the sum of his secret keys; i.e., $O_i = \sum_j K_{ij}$. The anonymous sender additionally adds his message M . This is illustrated in Figure 1.1(b).

Since $K_{ij} + K_{ji} = 0$, all secret keys cancel in the sum C , and only the message M remains. When several participants try to send a message during the same round, the messages collide (e.g. $C = M + M' + M''$) and no meaningful data is transmitted.

A major problem of the protocol is that no communication can take place if a malicious participant deliberately creates collisions all the time. As the anonymity of the honest participants must not be compromised, the detection of such a disruptor is difficult. While computationally secure variation have been proposed, no efficient and practically usable solution has been proposed for the information-theoretical setting until today.

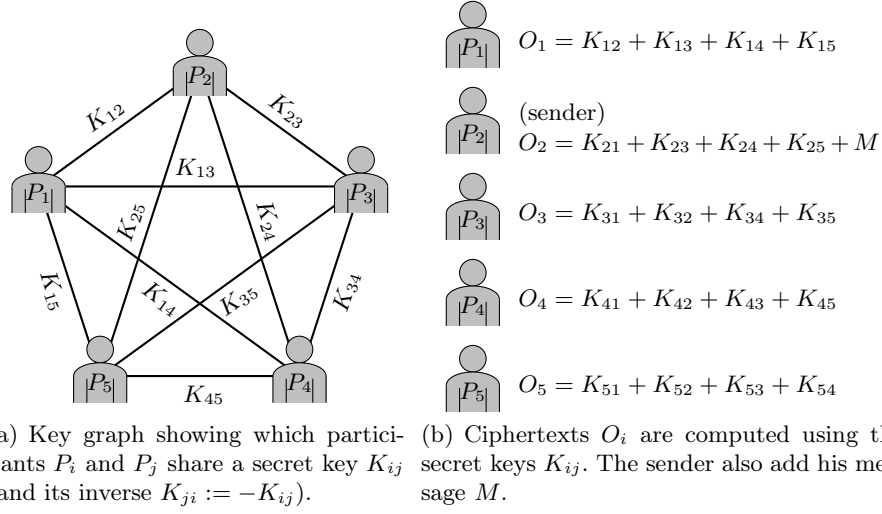


Fig. 1.1. Computation of ciphertexts in the dining cryptographers protocol.

Related Work In [6], Golle and Juels used the Diffie-Hellman key exchange to construct ciphertexts with an algebraic structure that can be used in zero-knowledge proofs. More recently, Franck showed in [5] how to use such ciphertexts to detect cheating participants in the context of collision resolution algorithms. However, this approach does not offer the unconditional anonymity of the initial protocol.

Our Contribution In this paper, we present a novel unconditionally untraceable dining cryptographers scheme with optimal collision resolution, in which computationally restricted disruptors are easy to detect. We use Pedersen commitments to computationally bind participants to their secret keys and then we use these commitments to prove the correct retransmission of messages in a tree based collision resolution algorithm (We use verifiable superposed receiving, presented by Pfitzmann in [9] and Waidner in [10]).

We believe our scheme is a significant improvement over the reservation based technique of the initial dining cryptographers protocol [4], wherein the detection of disruptors is lengthy and cumbersome. We see possible applications in various areas like low latency anonymous communication and electronic voting.

Outline of the Paper The rest of this paper is organized as follows. Section 2 contains the preliminaries. In Section 3, we show how we extend the dining cryptographers scheme with Pedersen commitments. In Section 4, we show how the commitments can be used to construct statements that can be used in zero-knowledge proofs. In Section 5, we show how to implement verifiable collision resolution. Section 6 contains some practical considerations. Section 7 is about related work, and Section 8 concluding remarks.

2 Preliminaries

In this section, we discuss the assumptions and the primitives that we use in the the rest of the paper.

Computational Assumptions We assume a group of n participants P_1, \dots, P_n that can be modeled by poly-time turing machines. We need a short-time computational assumption to verify the correct execution of the protocol in zero-knowledge. The anonymity of the transmitted data is unconditionally secure.

Communication Channels We assume a secure communication channel between each distinct pair of participants P_i and P_j , and we assume a fully connected key graph.

We further assume a reliable synchronous broadcast channel [7], which allows each participant $P_i \in \mathbb{P}$ to send a message to all other participants. The recipients thus have the guarantee that all then receive the same message, and that this is the same unfalsified message that was send out by the sender.

Pedersen Commitments [8] Let G be a group of order q in which the discrete logarithm problem is assumed to be hard, and let g and h be randomly chosen generators of a G . To commit to secret $K \in \mathbb{Z}_q$, the committer choses random $r \in \mathbb{Z}_q$ and computes the commitment

$$c := g^K h^r.$$

The committer can open the commitment by revealing (K, r) . Such a commitment is *unconditionally hiding*, which means that K is perfectly secret until the commitment is opened. Further, such a commitment is *computationally binding*, which means that it is computationally hard to find $(K', r') \neq (K, r)$, such that $c = g^{K'} h^{r'}$. And finally, such commitments are *homomorphic*; which means that for $c = g^K h^r$ and $c' = g^{K'} h^{r'}$, we also have $c'' = cc' = g^{K+K'} h^{r+r'}$.

Zero-Knowledge Proofs A zero-knowledge proofs allows a prover to convince a verifier that he knows a witness which verifies a given statement, without revealing the witness or giving the verifier any other information. One can for instance construct a zero-knowledge proof to show the knowledge of a discrete logarithm, the equality of discrete logarithms with different bases, and logical \wedge (and) and \vee (or) combinations thereof. A system for proving general statements about discrete logarithms was presented in [3]. In our notation based on [2], secrets are represented by greek symbols.

Example 1. A proof of knowledge of the discrete logarithm of y to the base g as

$$\mathcal{PK}\{\alpha : y = g^\alpha\}.$$

3 Extended Scheme with Pedersen Commitments

In this section, we propose a way to extend the dining cryptographers scheme using Pedersen commitments. We let each participant $P_i, i \in \{1, \dots, n\}$ broadcast a tuple $(O_i, c_i) \in \mathbb{Z}_q \times G$ instead of just broadcasting O_i . The element c_i is a Pedersen commitment to the value K_i . The algebraic (discrete log based) structure of c_i will later allow to prove statements about O_i in zero-knowledge. As c_i is unconditionally hiding, the security of the original protocol is preserved.

Detailed Description During the setup phase, when participants P_i and $P_j, i \neq j$ agree on a secret key $K_{ij} \in \mathbb{Z}_q$, we require them to additionally agree on a second secret value $r_{ij} \in \mathbb{Z}_q$. Similarly to $K_{ji} = -K_{ij}$, we define $r_{ji} = -r_{ij}$. To simplify the description we further define $r_{ii} = 0$. The value r_{ij} is then used by participant P_i to commit to the secret key K_{ij} , using the Pedersen commitment

$$c_{ij} := g^{K_{ij}} h^{r_{ij}}.$$

Note that P_i and P_j know the secrets r_{ij} and K_{ij} , so that both of them can compute and open c_{ij} . This knowledge is used by P_j to further provide P_i with a digital signature

$$\mathcal{S}_j(c_{ij}).$$

This digital signature can later be used by P_i to prove the authenticity of c_{ij} to a third party. Revealing c_{ij} and $\mathcal{S}_j(c_{ij})$ will not give away any information about K_{ij} , since c_{ij} is unconditionally hiding.

If participant P_i 's ciphertext O_i does not contain a message, we have

$$O_i = K_i$$

where

$$K_i := \sum_{j=1}^n K_{ij}.$$

A Pedersen commitment for K_i can be computed from c_{i1}, \dots, c_{in} according to

$$c_i := \prod_{j=1}^n c_{ij}.$$

This aggregation of commitments is illustrated in Figure 3.1, where P_1 computes the commitment c_1 for the ciphertext $O_1 = K_1$. This commitment c_1 could be opened by P_1 using K_1 and $\sum_{j=1}^n r_{1j}$.

During the broadcast phase, the participants P_1, \dots, P_n respectively send the tuples $(O_1, c_1), \dots, (O_n, c_n)$. The commitments c_1, \dots, c_n are valid if

$$\prod_{i=1}^n c_i = 1. \tag{3.1}$$

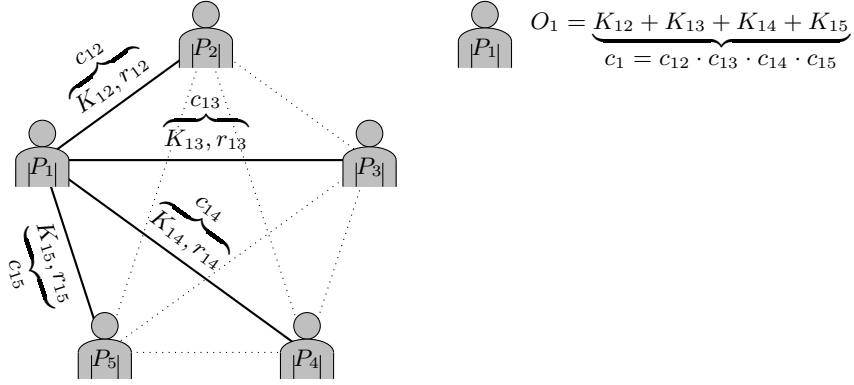


Fig. 3.1. Example: Binding to the secret keys using shared Pedersen commitments. Participant P_1 agrees on the secret keys $K_{12}, K_{13}, K_{14}, K_{15} \in \mathbb{Z}_q$ and the secret values $r_{12}, r_{13}, r_{14}, r_{15} \in \mathbb{Z}_q$ respectively with the participants P_2, P_3, P_4 and P_5 . This allows him to compute the commitments $c_{12}, c_{13}, c_{14}, c_{15} \in G$ with $c_{ij} = g^{K_{ij}} h^{r_{ij}}$. Finally, he computes $c_1 = \prod_{j=2}^5 c_{1j}$, the commitment for $K_1 := \sum_{j=2}^5 K_{1j}$.

If (3.1) does not hold, this means that at least one participant cheated. To find the cheater(s) an investigation phase can be performed.

During such an investigation phase, each participant P_i will publish the secret keys c_{ij} and the corresponding signatures $\mathcal{S}_j(c_{ij})$ for $j \in \{1, \dots, n\} \setminus \{i\}$. The signatures have to be correct, and it must hold that

$$c_i = \prod_{j=1}^n c_{ij} \quad (3.2)$$

and

$$c_{ij} c_{ji} = 1. \quad (3.3)$$

If a signature is wrong or if (3.2) or (3.3) does not hold, then the corresponding participant P_i cheated. The fact that (3.1) and (3.3) must hold is because we have $K_{ij} = -K_{ji}$ and $r_{ij} = -r_{ji}$ by construction, and (3.2) must hold by definition.

4 Statements for Zero-Knowledge Proofs

In this section, we propose statements that can be used in zero-knowledge proofs.

Statements about Single Rounds During a single round of the dining cryptographers protocol, a participant P_i broadcasts a ciphertext (O_i, c_i) . Either we have $O_i = K_i$ or $O_i = K_i + M$. A statement that holds when O_i does not encode a message M is given in Theorem 1. The proof is given in Appendix A.

Theorem 1. *If a poly-time participant P_i generates the tuple (O_i, c_i) and P_i knows α such that $c_i = g^{O_i} h^\alpha$, then we have $O_i = K_i$.*

Statements about Multiple Rounds In order to discuss multiple rounds, we use a superscript (k) to denote a value of a round k . E.g., the values $O_i^{(1)}$, $O_i^{(2)}$ and $O_i^{(3)}$ denote the ciphertexts broadcasted by P_i during the rounds 1, 2 and 3 respectively.

Theorem 2 provides a statement that holds when ciphertexts of two rounds encode the same message.

Theorem 2. *If a poly-time participant P_i generates the tuples $(O_i^{(1)}, c_i^{(1)})$ and $(O_i^{(2)}, c_i^{(2)})$, and P_i knows α such that $c_i^{(1)}(c_i^{(2)})^{-1} = g^{O_i^{(1)} - O_i^{(2)}} h^\alpha$, then $O_i^{(1)}$ and $O_i^{(2)}$ encode the same message.*

Theorem 3 provides a statement that holds when a message encoded in a first ciphertext is encoded at most once in a series of other ciphertexts (while the rest of the ciphertexts does not encode a message).

Theorem 3. *If a poly-time participant P_i generates $(O_i^{(1)}, c_i^{(1)}), \dots, (O_i^{(l)}, c_i^{(l)})$, and P_i knows α such that*

$$\bigwedge_{k=2}^l \left(\left(c_i^{(1)} \prod_{j=2}^k (c_i^{(j)})^{-1} = g^{O_i^{(1)} - \sum_{j=2}^k O_i^{(j)}} h^\alpha \right) \vee \left(c = g^{O_i^{(k)}} h^\alpha \right) \right) \quad (4.1)$$

then at most one ciphertext of $O_i^{(2)}, \dots, O_i^{(l)}$ encodes the same message as $O_i^{(1)}$, while the other ciphertexts of $O_i^{(2)}, \dots, O_i^{(l)}$ encode no message.

The statements from the preceding theorems can be used in zero-knowledge proofs. We will see in the next section how we can use this for proving the correct execution of a collision resolution algorithm.

5 Implementing Verifiable Superposed Receiving

Superposed receiving is a collision resolution scheme for the Dining Cryptographers protocol proposed by Pfitzmann in [9] and Waidner in [10]. It achieves an optimal throughput of one message per round. However, the scheme was never used in practice, as a malicious participant may disrupt the process and remain undetected. In this section, we show that in our scheme such disruptors are easy to detect.

Superposed Receiving A collision occurs when multiple participants send a message in the same round. In superposed receiving, collisions are repeatedly split in two, until all messages are transmitted. An exemplary collision resolution tree is shown in Figure 5.2. To keep our description simple, we assume that when a collision occurs in a round k , the rounds $2k$ and $2k + 1$ are used to split this collision. Like in the previous section, we use subscripts to denote values of the different rounds, e.g. $O^{(7)}$ for ciphertext of round 7.

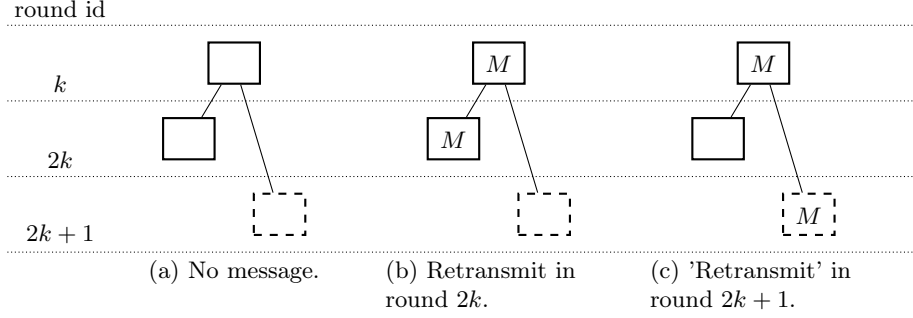


Fig. 5.1. Retransmission in superposed receiving. Only message involved in the collision in round k may be retransmitted either in round $2k$. No new message may be sent during the collision resolution process.

In superposed receiving messages are tuples of the form $(1, m)$. It is then possible to determine the number of messages involved in a collision and to compute the average value of a message involved in the collision. For instance, when 3 messages $(1, m)$, $(1, m')$ and $(1, m'')$ collide in round k , the tuple $(3, m + m' + m'')$ is received and the average value is then $(m + m' + m'')/3$. Then, in round $2k$ only the messages with a value below this average are retransmitted, while the rest of the messages goes to round $2k + 1$. To keep our description simple, we assume that a tuple of the form $(1, m)$ is encoded in a message $M \in \mathbb{Z}_q$, such that the individual elements of the tuple are added when there is a collision.

It is not necessary to transmit anything in round $2k + 1$. Instead, the result of round $2k + 1$ is inferred by subtracting the results of round $2k$ from the result of round k . This technique, which is also known as inference cancellation [11], is the reason for the optimal throughput of the scheme. In the example of Figure 5.2 only 5 rounds are transmitted for 5 messages. For inference cancellation to work, the collision resolution must operate in blocked access mode. This means that no new message may be sent by any participant until the collision resolution process is over.

Verification A malicious participant may try to disrupt the collision resolution process by not properly participating in the collision resolution process. We verify the correct execution of the protocol in two steps.

First, we verify in round $2k$ that, according to Figure 5.1, a participant either retransmits exactly the same message that he sent in round k , or that he sends no message at all. Using the statements from the previous section, each participant can prove that his ciphertext O_{2k} is correct, without revealing if whether it contains a message or not. To do this, the participant generates a zero-knowledge proof that proves that he knows α such that

$$\mathcal{PK}\{\alpha : (c_i^{(2k)} = g^{O_i^{(2k)}} h^\alpha) \vee (c_i^{(k)} c_i^{(2k)-1} = g^{O_i^{(k)} - O_i^{(2k)}} h^\alpha)\} \quad (5.1)$$

holds. With this proof he can convince a verifier that he participated correctly, without compromising the anonymity of the protocol. As described before, in some rounds no transmission takes place and so there might not be a value $O_i^{(k)}$ available to prove the correctness of $O_i^{(2k)}$ using statement (5.1). However, it is still possible to prove that $O_i^{(2k)}$ is correct by proving that a message contained in the nearest transmitted parent round is transmitted at most once in all the branches down to $O_i^{(2k)}$. This can be done using Theorem 3.

Example 2. In the collision resolution process shown in Figure 5.2, a participant proves for $O^{(2)}$ that

$$\mathcal{PK}\{\alpha : (c_i^{(2)} = g^{O_i^{(2)}} h^\alpha) \vee (c_i^{(1)} c_i^{(2)-1} = g^{O_i^{(1)} - O_i^{(2)}} h^\alpha)\}$$

holds, then for $O^{(4)}$ that

$$\mathcal{PK}\{\alpha : (c_i^{(4)} = g^{O_i^{(4)}} h^\alpha) \vee (c_i^{(2)} c_i^{(4)-1} = g^{O_i^{(2)} - O_i^{(4)}} h^\alpha)\}$$

holds, then for $O^{(6)}$ that

$$\mathcal{PK}\{\alpha : (c_i^{(6)} = g^{O_i^{(6)}} h^\alpha) \vee (c_i^{(1)} c_i^{(2)-1} c_i^{(6)-1} = g^{O_i^{(1)} - O_i^{(2)} - O_i^{(6)}} h^\alpha)\}$$

holds, then for $O^{(14)}$ that

$$\mathcal{PK}\{\alpha : (c_i^{(14)} = g^{O_i^{(14)}} h^\alpha) \vee (c_i^{(1)} c_i^{(2)-1} c_i^{(6)-1} c_i^{(14)-1} = g^{O_i^{(1)} - O_i^{(2)} - O_i^{(6)} - O_i^{(14)}} h^\alpha)\}$$

holds.

This shows that it is possible to verify that a participant retransmitted his message in only one branch of the tree.

Then, we verify that every properly collision splits into 2 parts. As we know that every collision is supposed to split, we know if all messages end up in the same branch then at least one participant cheated. So this can only happen when a malicious node retransmits message in the wrong branch, or when the message is not of the correct form $(1, m)$ initially. If such activity is detected, it is possible to identify the disruptor by falling fall back to probabilistic splitting of collisions [5,11]. Each participant then choses randomly whether to retransmit his message in round $2j$ or round $2j + 1$. This allows to separate the honest nodes from the malicious ones after a few rounds. After this separation has taken place it is possible to determine the messages that have not been transmitted in the right branch earlier, and to identify the corresponding participants using a zero-knowledge proof (I.e. each participant has to prove in zero-knowledge that he did not send the message that appeared in the wrong branch.). If a collision repeatedly does not split, even with probabilistic retransmission, then the involved participants can be considered to be malicious.

A disruptor will thus always be detected and can be banned from the group of participants.

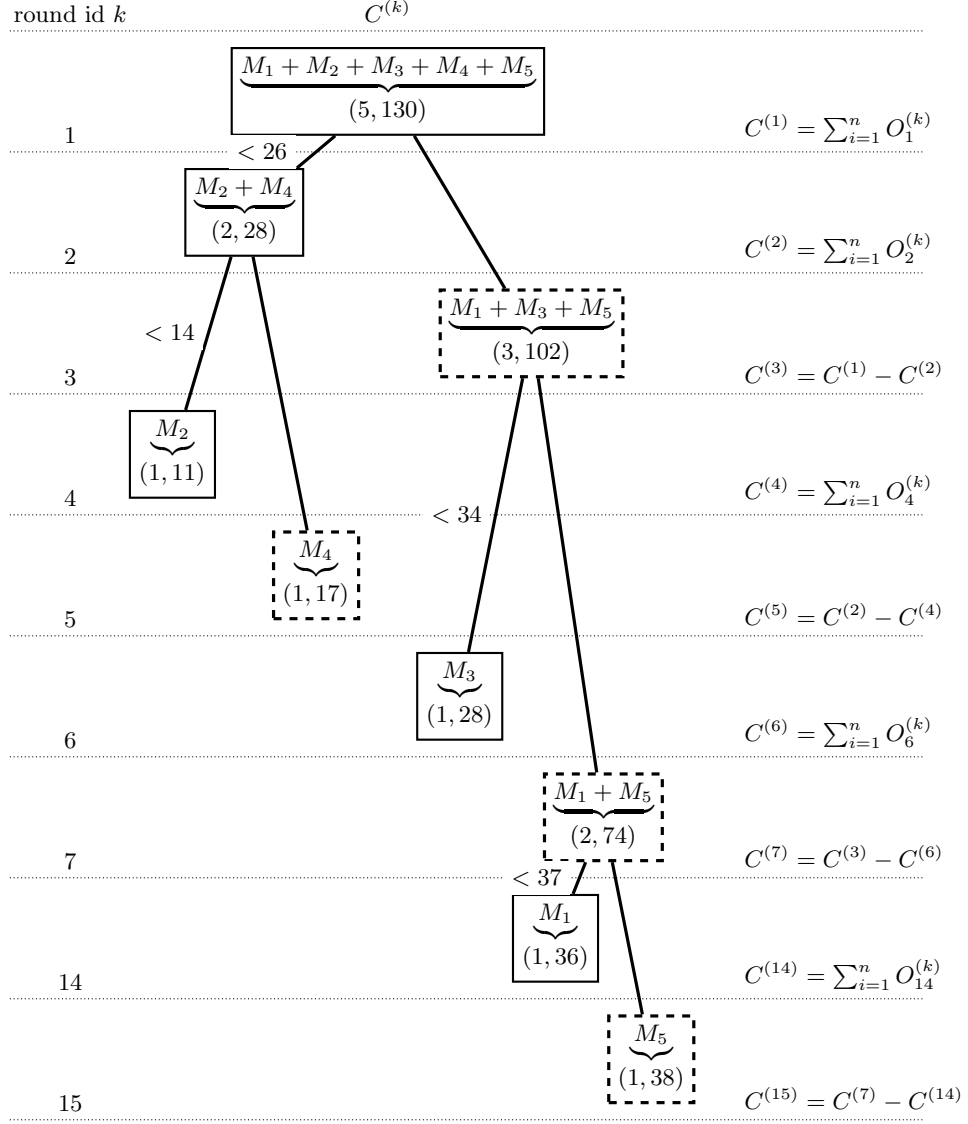


Fig. 5.2. Exemplary binary collision resolution tree with superposed receiving. In rounds 1,2,4,6 and 14, ciphertexts $O^{(k)}$ are transmitted, and $C^{(k)}$ is computed using these ciphertexts. In rounds 3,5,7 and 15, no data is transmitted and $C^{(k)}$ is computed using data from the parent and the sibling node.

6 Practical Considerations

This section contains a few remarks about various aspects of a practical implementation.

(Signatures with Merkle Trees) During the setup, participants mutually authenticate their commitments c_{ij} and c_{ji} using the signatures $\mathcal{S}_j(c_{ij})$ and $\mathcal{S}_i(c_{ji})$. For many rounds, this can be implemented more efficiently using a Merkle tree.

(Mutual Signatures Attack) During the setup phase one participant could refuse to agree on a shared secret with another participant. I.e., one participant could refuse to provide a signature $\mathcal{S}(\cdot)$ to the other participant. As a response to this, we suggest that each participant may just publicly claim that he is not sharing a secret key with the other participant. It is then assumed the corresponding $K_{ij} = 0$ and $c_{ij} = 1$, and no signature is required.

(Efficient Investigation in Packet-Switched Networks) In order to efficiently detect disruptors, a single (trusted) investigator can collect and verify the proofs. Only when he detects a cheater, he will provide all other participants with a copy of the relevant data.

(Long Messages) To keep the description simple, we assumed that messages fit in a single element of \mathbb{Z}_q ; i.e., that a single K is sufficient for one round. For longer messages one can, as shown in [1], use a randomly chosen generator tuple g, g', g'', \dots, h to commit to a vector (K, K', K'', \dots) by computing

$$c = g^K g'^{K'} g''^{K''} \dots h^r.$$

(Key Establishment) To obtain information-theoretical security from the protocol, it is necessary to use real random secret keys for each round. In practice, it is also possible to realize weaker system, where the shared secrets are generated for instance using the Diffie-Hellman protocol.

7 Concluding Remarks

We have shown how to extend the dining cryptographers scheme with Pedersen commitments, such that it is possible to construct zero-knowledge proofs about the retransmission of data, without compromising the anonymity of the protocol.

It is remarkable that it is then possible realize a verifiable dining cryptographers protocol with an optimal throughput, which does not require any kind of reservation phase prior to the transmission of the messages.

We believe that our approach is a significant step forward towards the efficient implementation of unconditionally untraceable communication systems.

We see possible applications in many fields, like low-latency untraceable communication and secret shuffling. The main problem that remains in practice is the secure agreement on secret keys between participants.

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A Proofs

Lemma 1. *Given randomly chosen generators g, h of a group in which the discrete log problem is assumed to hold, a poly-time participant can only find $(a, b), (a', b')$, such that when $g^a h^b = g^{a'} h^{b'}$, it must hold that $a = a'$.*

Proof. If a poly-time participant can find $(a, b), (a', b')$ such that $g^a h^b = g^{a'} h^{b'}$ with $a \neq a'$, then he can also compute the discrete logarithm $\log_h g$ with

$$\log_h g = (b' - b)/(a - a').$$

As this is impossible by assumption, the statement follows. \square

Theorem 4. *If a poly-time participant P_i generates the tuple (O_i, c_i) and P_i knows α such that $c_i = g^{O_i} h^\alpha$, then we have $O_i = K_i$.*

Proof. By definition, we have $c_i = g^{K_i} h^{r_i}$. If poly-time participant P_i knows O_i and α , such that $c_i = g^{O_i} h^\alpha$, it follows from Lemma 1 that $O_i = K_i$. \square

Theorem 5. *If a poly-time participant P_i generates the tuples $(O_i^{(1)}, c_i^{(1)})$ and $(O_i^{(2)}, c_i^{(2)})$, and P_i knows α such that $c_i^{(1)}(c_i^{(2)})^{-1} = g^{O_i^{(1)} - O_i^{(2)}} h^\alpha$, then $O_i^{(1)}$ and $O_i^{(2)}$ encode the same message.*

Proof. By definition, we have $O_i^{(2)} = K_i^{(2)} + M_a$ and $O_i^{(1)} = K_i^{(1)} + M_b$, where $M_a, M_b \in \mathbb{Z}_q$. Further, we have

$$\begin{aligned} c_i^{(1)}(c_i^{(2)})^{-1} &= g^{O_i^{(1)} - O_i^{(2)}} h^\alpha \\ g^{K_i^{(1)} - K_i^{(2)}} h^{r_i^{(1)} - r_i^{(2)}} &= g^{O_i^{(1)} - O_i^{(2)}} h^\alpha. \end{aligned}$$

According to Lemma 1 it follows that $K_i^{(1)} - K_i^{(2)} = O_i^{(1)} - O_i^{(2)}$ and thus

$$M_a = M_b,$$

which is the statement. \square

Theorem 6. *If a poly-time participant P_i generates $(O_i^{(1)}, c_i^{(1)}), \dots, (O_i^{(l)}, c_i^{(l)})$, and P_i knows α such that*

$$\bigwedge_{k=2}^l \left(\left(c_i^{(1)} \prod_{j=2}^k (c_i^{(j)})^{-1} = g^{O_i^{(1)} - \sum_{j=2}^k O_i^{(j)}} h^\alpha \right) \vee \left(c_i = g^{O_i^{(k)}} h^\alpha \right) \right) \quad (\text{A.1})$$

then at most one ciphertext of $O_i^{(2)}, \dots, O_i^{(l)}$ encodes the same message as $O_i^{(1)}$, while the other ciphertexts of $O_i^{(2)}, \dots, O_i^{(l)}$ encode no message.

Proof. With $c_i := g^{K_i} h^{r_i}$ and Lemma 1 it follows that when (A.1) holds, we have

$$\bigwedge_{k=2}^l \left(\left(O_i^{(1)} - K_i^{(1)} = \sum_{j=2}^k O_i^{(j)} - K_i^{(j)} \right) \vee (O_i^{(k)} = K_i^{(k)}) \right).$$

Assume the ciphertext O_1 encodes the message M , so that $O_1 = K_1 + M$ (with possibly $M = 0$). For $k = 2$, we can then have either $O_k = K_k + M$ or $O_k = K_k$. For $k > 2$ and $\sum_{j=2}^{k-1} O_j - K_j = 0$, we can have either $O_k = K_k + M$ or $O_k = K_k$. For $k > 2$ and $\sum_{j=2}^{k-1} O_j - K_j = M$, we must have $O_k = K_k$. That is, for increasing k , as long as O_2, \dots, O_{k-1} contains no message, we can have either $O_k = K_k + M$ or $O_k = K_k$. Once one ciphertext of O_2, \dots, O_{k-1} contains the message M , we must have $O_k = K_k$. Thus, at most one ciphertext of O_2, \dots, O_l may encode the message M encoded in O_1 , while the other ones contain no message, which is the statement. \square